Stability and Exchange in a Generalized Diamond-Dybvig Model

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Abstract

The recent literature on bank runs, following Diamond and Dybvig (1983), studies the banking sector in isolation from the greater economy. Here we model an economy that includes not only DD type bank depositors but also producers of goods. When consumers can exchange goods for deposits, trade provides a welfare improvement, and bank runs are not an equilibrium unless the bank is fundamentally insolvent. Instability is, thus, not inherent to banking but results from restrictions on information and exchange. This is consistent with historical evidence from the US banking system.

1 Introduction

Diamond and Dybvig (1983) is by far the most commonly cited article on the theory of deposit banking. The DD model illustrates the important role of banks as providers of liquidity but also the danger they face due to the potential for runs. The authors discuss suspension of convertibility in optimal deposit contracts and the potential role of government in providing deposit insurance. Many citations of DD (1983) regard it as describing the inherent instability of deposit banking. Here, by contrast, we show that instability is a special case of the model rather than a general property.

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1DD (1983) has 861 citations according to the Social Science Citation Index as of March 5, 2011. It is ranked as the 24th highest paper by impact factor by Research Papers in Economics (http://repec.org).
Dozens of papers have extended the DD model in various ways. Optimal deposit contracts have been discussed by Wallace (1990), Villamil (1991), Selgin (1993), Peck and Shell (2003), and Green and Lin (2003). Many have debated the presence or absence of bank run equilibria including Postlewaite and Vives (1987), Hazlett (1997), Green and Lin (2000), Goldstein and Pauzer (2005), and Ennis and Keister (2009). Further extensions of the DD model include Chari (1989), Dowd (1993), Wallace (1996), Cooper and Ross (2002), and Cavalcanti (2004). Garratt and Keister (2005), Schotter and Yorulmazer (2003), and Duffy (2008) describe experimental tests of the DD model. Yet none of these works consider the banking system in the context of a larger economy. Each assumes that banking occurs in isolation rather than interacting with other sectors as originally proposed in DD (1983). Wallace (1990, p.8) suggests that with the sequencal service constraint taken seriously, “The model suggests that illiquid banking system portfolios can be understood without complications like separate financial and business sectors.” All progressive works appear to have followed on that assumption.

This paper demonstrates that the banking sector cannot be effectively studied in isolation. It explores a state of the DD model which includes both banking and production sectors. DD (1983, p.409) briefly discusses this scenario but does not explore it in detail. There are two benefits to the multi-sector model. First, agents can hedge risk by investing in multiple sectors which creates improvements in welfare. The same risk aversion that motivates agents to invest in a bank also motivates them to invest outside the bank. Second, exchange can prevent bank runs because deposit contracts can be traded at a discount to real goods. As the bank nears default, the uncertain payoff will cause deposits to be traded at a price lower than the stated value but higher than the default value. The likelihood of bank runs is thereby reduced and is not an equilibrium unless the bank is fundamentally insolvent. This is consistent with historical experience of US banking before the Federal
Reserve. When a bank was suspected of insolvency, its notes continued to circulate but were often traded at a discount to their redemption value.

The issue of bank stability is especially pertinent today due to the current turmoil in the financial sector. Uhlig (2010) uses the DD model to describe the financial panic of 2008 as a modern-day bank run. Other works use the DD model to analyze instability and contagion in the financial sector (Manz 2010, Ennis and Keister 2010, and Peck and Shell 2010). Yet if analysis of the banking sector in isolation is inappropriate, as this paper suggests, then the use of standard DD models to study these phenomena will draw precisely the wrong conclusions about bank stability, especially regarding the degree to which government intervention can stem financial panics.

In a multi-sector DD model solvent banks do not experience runs. The next section describes the basics of the DD model and outlines the variables which influence stability and exchange. Section 3 describes the incentives for trade in a multi-sector economy and their effects on bank stability. Section 4 discusses historical evidence from the US banking system. Section 5 concludes with implications for further research.

2 The Diamond-Dybvig Model

This section will describe the basic model described in DD (1983). Although the reader may be familiar with the DD model, we discuss it here for two reasons. First, we outline the incentives for both production and banking sectors so that they may be easily combined in section 3. Second, we emphasize the importance of analyzing ex post consumption. While most DD models focus on ex ante optimization in period 0, the possibility of a bank run can be better analyzed ex post in period 1 after agents have learned their types. This is also the period when exchange, if any, will occur.

We begin by describing the agent population and available technology in section 2.1.
Section 2.2 calculates ex ante optimal consumption which is shown to be beyond the limit of the basic technology. The formation of banks is described in section 2.3 and the potential for runs in 2.4. Section 2.5 calculates actual ex post consumption, and section 2.6 shows that banks may still be created despite the threat of a run.

2.1 The All-Production Economy

Consider an economy to be analyzed over 3 time periods \((T = 0, 1, 2)\). Each agent of the population is endowed with 1 unit of capital at time \(T = 0\). The economy has only one production technology with the following properties:

\[
\begin{array}{c|cc}
T=0 & T=1 & T=2 \\
\hline
-1 & 0 & R \\
1 & 0 & 1
\end{array}
\]

A capital investment of 1 unit at \(T = 0\) produces consumption goods of up to 1 unit at \(T = 1\), up to \(R > 1\) units at \(T = 2\), or any linear combination thereof. At \(T = 1\) each agent allocates consumption for periods 1 and 2. Individual consumption is not publicly observable.

There are two types of agents, patient and impatient. As of \(T = 0\), agent type is unknown, and all agents identically prefer to patiently defer consumption until period \(T = 2\). However, each faces some positive probability of becoming impatient in period \(T = 1\), in which case he will chose to maximize consumption in that period. Impatient agents who prefer consumption at \(T = 1\) are described as type 1 with \(\Theta = 1\), while patient agents who prefer consumption at \(T = 2\) are type 2 with \(\Theta = 2\). Agent types are also not publicly observable, however the proportion \(t \in (0, 1)\) of type 1 agents in the population of \(N\) agents where \(t = \Sigma_{i=1}^{N} (2 - \Theta) / N\) is known to all agents.
Let $c^\Theta_T$ represent individual consumption by type $\Theta$ agents in period $T$.\footnote{Notations of $c_1$ and $c_2$ refer to consumption in periods 1 and 2 regardless of agent type. Alternatively, $c^1$ and $c^2$ represent consumption for agent types 1 and 2 regardless of time.} Impatient type 1 agents always maximize consumption in period 1. Patient type 2 agents can maximize consumption in period 2 either by leaving their capital in the production process until $T = 2$ or by withdrawing consumption goods in period 1 and storing them at no cost for consumption at $T = 2$. All agents exhibit constant relative risk aversion in utility of consumption with $0 < \gamma < 1$. Type 2 agents face some discount factor $0 < \rho < 1$. These utilities are given by equation 2.1.

$$u(c_1, c_2, \Theta) = \begin{cases} \frac{(c^\Theta_1)^{1-\gamma}}{1-\gamma} & \text{if } \Theta = 1 \\ \rho \left(\frac{c^\Theta_1 + c^\Theta_2}{1-\gamma}\right)^{1-\gamma} & \text{if } \Theta = 2 \end{cases}$$ (2.1)

Individual utility can be maximized subject to the resource constraint 2.2.

$$c_2 = R(1 - c_1)$$ (2.2)

These preferences indicate that each agent will invest his endowment of 1 unit at $T = 0$. When agent types are revealed at $T = 1$, all type 1 agents will maximize consumption in $T = 1$. All type 2 agents will defer consumption to maximize at $T = 2$. The price of $c_2$ in terms of $c_1$ will be $R^{-1}$. Agents may exchange promises of goods in different periods, but as DD describe, “Given these prices, there is never any trade” (DD 1983, p.406). Therefore, all type 1 agents will withdraw and consume 1 unit at $T = 1$, while all type 2 agents will refrain until $T = 2$ when they receive and consume $R$ units.

These choices are presented graphically in figure 1. The resource constraint, drawn in solid black, is the same for both agent types. Gray lines represent the indifference curves for each agent, the points at which the agent is indifferent between consumption at times 1 and
2. Type 1 agents have vertical indifference curves since they consume only $c_1$. Indifference curves are unitary for type 2 agents since $c_1$ and $c_2$ are perfect substitutes. Thus, type 1 agents maximize their consumption set $\bar{c}^1 = (1, 0)$ of 1 unit at time $T = 1$ and 0 time $T = 2$. Type 2 agents leave their goods in production and consume $\bar{c}^2 = (0, R)$ of nothing at time $T = 1$ and $R$ units at time $T = 2$.

2.2 Optimal Consumption

Expected individual consumption can be shown to be below the social optimum. Social welfare in the all-production economy can be stated as the weighted sum of utilities by agent type shown in equation 2.3.

$$W_{Prod} = tu(c_1^1) + (1 - t)\rho u(c_1^2 + c_2^2)$$ (2.3)

Social welfare can be maximized according to the constraints on resources and between marginal production and consumption.

$$tc_1^1 + \frac{(1 - t)c_2^2}{R} = 1$$ (2.4)
\[ u'(\hat{c}_1) = R u'(\hat{c}_2), \quad \hat{c}_2 = \hat{c}_1^2 = 0 \]  

Optimal individual consumption in each period can be calculated as equations 2.6 and 2.7.\(^3\)

\[ \hat{c}_1 = \frac{R}{tR + (1 - t)(\rho R)^{1/\gamma}} \]  

\[ \hat{c}_2 = \frac{\rho R^{1/\gamma}}{tR + (1 - t)(\rho R)^{1/\gamma}} \]  

Because agents are risk averse, their optimal consumption is some combination of \(c_1\) and \(c_2\). However, agents acting independently are unable to achieve this socially optimal consumption set. As DD (1983, p.407) describes “Optimal consumption levels satisfy \(\hat{c}_1 > 1\) and \(\hat{c}_2 < R\). Therefore, there is room for improvement on the competitive outcome.”

2.3 The All-Banking Economy

In the DD model, it is sometimes possible to improve upon the competitive equilibrium through collective action.\(^4\) At \(T = 0\), each agent can increase his expected future utility by trading off some potential consumption at \(T = 2\) for increased consumption at \(T = 1\).

Suppose that all agents agree to pool their capital investments as a form of insurance against the lower utility that comes with being impatient. Each agent receives a deposit contract which he may redeem at \(t = 1\) for an amount \(r_1 \geq 1\). Any goods not withdrawn at \(T = 1\) will be used to produce consumption goods which will be divided equally among the remaining agents at \(T = 2\). Return \(r_2\) at time 2 is, therefore, a function of \(t\).

\[ r_2(t) = \frac{(1 - tr_1)R}{1 - r_1}. \]  

\(^3\)These calculations are shown in Appendix A.

\(^4\)It is not always possible to improve upon the no-trade equilibrium since in some cases the potential for runs may preclude any gains, so no bank will be established. This possibility is acknowledged by DD (1983, p.409), expanded by Huo and Yu (1994), and will be discussed further in section 2.6.
To optimize expected utility as of $T = 0$, the group sets $r_1 = \hat{c}_1$ from equation 2.6. Expected time 2 consumption will be the optimal value $r_2(t) = \hat{c}_2$ from equation 2.7. This consumption set increases social welfare above the level achieved in the all-production economy.

### 2.4 Bank Runs

The optimal levels of consumption described in the previous section will only result if information about agent types is publicly observable. Banks can ensure optimal risk sharing by verifying individual agent type $\Theta$. Alternatively, they may use other mechanisms such as suspension clauses based on the portion $t$ of type 1 agents (DD 1983, p.411). However, when neither individual agent type nor the proportion of type 1 agents is observable, consumption may be suboptimal since some type 2 agents may choose to consume at $T = 1$. When this information is lacking, “No bank contract can attain the full-information optimal risk sharing” (p.412).

Suppose now that $t$ is random and stochastic. Agents have only some expectation $\hat{t}$ which they use to calculate optimal consumption. If $t \neq \hat{t}$, consumption is suboptimal. For all $t < \hat{t}$, first period consumption will be too low, and when $t > \hat{t}$ period 1 consumption will be too high and consumption in period 2 too low. In fact, consumption in time $T = 1$ may be so high that there are no goods are left for consumption in $T = 2$. In this case, the bank becomes insolvent before the second period. The threshold for insolvency is given in equation 2.9.

$$r_1(t + e_1^2) > 1.$$  \hspace{1cm} (2.9)

When period 1 consumption is expected to be above the threshold, all agents have the incentive to redeem their deposits at $T = 1$ which constitutes a bank run.

When a run occurs, all agents attempt to redeem their deposits at $T = 1$, however,
not all will be successful. We follow DD (1983, p.408) in assuming a sequential service constraint. All agents who choose to redeem at $T = 1$ form a line at the bank, and each agent’s place in line is assigned at random. The line progresses with each agent redeeming his deposit for $r_1$ units until the bank is devoid of capital. Thus, each of the first $1/r_1$ redeemers will receive $r_1$ units of consumption goods from the of 1 total unit of invested capital (representing 100% of the capital in the economy). The remaining $1 - 1/r_1$ agents are left with nothing. Therefore, each agent’s welfare in the case of a run is a probabilistic outcome shown in equation 2.10.

$$W_{\text{Run}} = (1/r_1)u(r_1) + (1 - 1/r_1)u(0)$$ (2.10)

Welfare in the bank run equilibrium is clearly lower than the no-run equilibrium and even lower than in the all-production economy. Total social welfare in the all-bank economy can be calculated as the weighted expected value of run and no-run equilibria given some expected probability $\delta$ that a run will occur.

$$W_{\text{Bank}} = (1 - \delta)W_{\text{NoRun}} + \delta W_{\text{Run}}$$ (2.11)

The incentive to run on the bank can be described in terms of the period 1 value $V(c_1)$ of deposits to each type of depositor as illustrated in figure 2. These graphs show the deposit value on the y-axis as a function of time 1 redemptions $c_1 = c_1^1 + c_1^2$ on the x-axis. Figure 2a. depicts the “fundamental” value of the deposit earned if the deposit is redeemed in the period which matched the agent’s type. Type 1 depositors always redeem in $t = 1$. They earn $c_1^1 = r_1$ up to the point where there $c_1 = 1 - t$ redemptions, at which time the bank defaults. After the point of default, all agents have an expected payoff of $r_1$ with probability $1/r_1$ and a payoff of 0 with probability $1 - 1/r_1$. This probabilistic default
payoff, denoted here as \( r_0 \), can range anywhere from \(-\infty < r_0 < 1/r_1\) depending on the values of \( \rho \) and \( \gamma \) but for convenience is shown in the range \( 0 < r_0 < 1 \).

The fundamental value of deposits to type 2 agents \( r_2 \) is a function of \( c_1 \) given in equation 2.8. This is shown in figure 2a. as a line declining from \( R \) at \( c_1 = 0 \) to 1 at \( c_1 = 1 - t \), the point where the bank defaults. When the bank goes into default, even type 2 agents choose to redeem their deposits since nothing will be left at \( T = 2 \). All deposits will be worth \( r_0 \) regardless of agent type. However, type 2 agents may decide to redeem their deposits at \( T = 1 \) even before \( c_1 = 1 - t \). Figure 2a. shows that after the point \( c_1 = 1/r_1 - t \), the time \( T = 2 \) redemption value falls below the value at \( T = 1 \) to \( r_2 < r_1 \). Since type 2 agents value \( c_1^2 \) and \( c_2^2 \) equally, they will choose to redeem their deposits early at \( T = 1 \). Since all agents decide to redeem at \( T = 1 \), there is a run on the bank, and the value of deposits falls to \( r_0 \). This scenario is depicted in figure 2b. Whenever the expected value of \( c_1 > 1/r_1 - t \), all agents redeem at \( T = 1 \) causing a run on the bank, and the expected value of all deposits falls to \( r_0 \).

Runs can occur for two reasons. First, as previously described, the number of types 1 agents may be higher than expected to the point that they consume all goods at \( T = 1 \),
and none are left for consumption at $T = 2$. This will be referred to as a “fundamental run” since the bank has more legitimate claims than it can pay out. The second type occurs when type 2 agents choose to consume at $T = 1$. This “sunspot run” can occur if the agents fear that the number of type 1 agents may be higher than previously expected (that $t > \hat{t}$). This differs from the fundamental run since it is only the expected difference between $t$ and $\hat{t}$ that causes the run.

What causes this shift in expectations? DD do not specifically say. It may be any economy-wide signal or event and need not be economic in nature. As described by DD (1983, p.410), it “need not be anything fundamental about the bank’s condition” but could be any “commonly observed random variable in the economy... even sunspots.” In this case, even a solvent bank will experience a run and default. In fact, any widely observable signal may cause all banks in the economy to be run upon simultaneously.

Although the information which can signal the run is not specifically included in the DD model, other works have provided explicit mechanisms. Green and Lin (2000) assumes that information is revealed by the agent’s order in line at the bank. Alonso (1993) uses a signal from each agent that may or may not be truthful, while Andolatto, Nosal, and Wallace (2007) assumes that agents reveal their true types when redeeming deposits. In Samartin (2003), agents observe their return on investment from which they can deduct the portion of impatient agents. Because this paper attempts to replicate DD (1983) as originally written, we continue on the assumption that agents receive some new information at $T = 1$ which is not included in the model.

2.5 Actual Consumption

We can study actual consumption (as opposed to ex ante expected consumption) by analyzing agents’ consumption opportunities at $T = 1$. Because type 1 agents only get utility
from consumption at time 1, we assume that all \( t \) agents will maximize period 1 consumption \( c_1^1 = r_1 \) with \( c_1^2 = 0 \). Therefore, type 2 agents will receive all of the remaining \( 1 - tr_1 \) capital which they will divide between \( c_2^1 \) and \( c_2^2 \) consumption at times 1 and 2.

The actual consumption in period 2 is a non-linear function of consumption in period 1. In equation 2.8, \( r_2 \) is dependent on period 1 consumption by type 1 agents. However, since some type 2 agents may also consume in period 1, actual period 2 consumption by type 2 agents \( c_2^2 \) is dependent on \( t + c_1^2 \) as shown in equation 2.12.

\[
c_2^2 = \frac{R(1 - r_1(t + c_1^2))}{1 - t - c_1^2}
\] (2.12)

The more agents who redeem their deposits at \( T = 1 \), the lower the payoff to patient depositors who withdraw at \( T = 2 \). The number of early redeemers can increase until the point \( c_1^2 = 1/r_1 - t \) where all resources are consumed, the bank becomes bankrupt, and a run occurs. Since deposits will be worth nothing at \( T = 2 \), all type 2 agents will choose to redeem their deposits at \( T = 1 \). Again, \( c_1^2 \) need not actually reach the level of \( c_1^2 = 1/r_1 - t \). Merely the expectation that this will occur is sufficient to create a bank run.

We can optimize social welfare according to the bank’s resource constraint. The welfare function in equation 2.3 includes only type 2 depositors. Type 1 agents care only for consumption at \( T = 1 \) and uniformly choose \( c_1^1 = r_1 \). Therefore, the social welfare function given in equation 2.13 depends solely on consumption by type 2 agents.

\[
W_{Bank} = tu(r_1) + (1 - t)\rho u(c_1^2 + c_2^2)
\] (2.13)

Unlike consumption in the all-production economy shown in figure 1, the resource constraint is not a straight line. While total period 1 consumption is less than \( c_1 = 1/r_1 - t \), individual period 2 consumption is a function of \( c_1 = t + c_1^2 \). Once total period 1 consump-
tion reaches \( c_1 = 1/r_1 - t \), the bank defaults, and \( c_2 = 0 \) as shown in 2.14.

\[
c_2^2 = \begin{cases} 
\frac{R(1-r_1(t+c_1^2))}{1-t-c_1} & \text{for } c_1^2 < 1/r_1 - t \\
0 & \text{for } c_1^2 \geq 1/r_1 - t 
\end{cases}
\]  

Figure 3 shows the resource constraints between \( c_1^2 \) and \( c_2^2 \) and utility functions for agent types 1 and 2. This analysis matches the proof in DD (1983) that banks are subject to non-fundamental runs, however, it deviates from their analysis in another respect. DD (1983) analyzes ex ante expected consumption as of \( T = 0 \). It is assumed that actual consumption in the non-run equilibrium will match the optimal levels predicted at that time. However, that prediction does not match the agents’ ex post decisions analyzed here as of \( T = 1 \). As shown in figure 3, type 2 agents do not optimize consumption by redeeming only at \( T = 2 \). Rather, they consume some combination of \( c_1^2 \) and \( c_2^2 \). This is due to the nonlinear nature of \( r_2 \) as a function of \( c_1 \). By redeeming a small amount of their deposits at \( T = 1 \) for \( r_1 \), they reduce the number of depositors waiting to redeem at \( T = 2 \). In the second period, goods are divided among as smaller group, so each agent gets a larger
share. The optimal consumption for type 2 agents is shown below.\(^5\)

\[
\hat{c}_1^2 = 1 - t - \sqrt{R(r_1 - 1)} \tag{2.15}
\]

\[
\hat{c}_2^2 = \frac{1 - r \left(1 + \sqrt{R(r_1 - 1)}\right)}{r - 1} \tag{2.16}
\]

At this consumption set, period 1 consumption is always below the bank run threshold

\[
c_1^2 < \frac{1}{r_1 - t}. \tag{2.17}
\]

Thus, there exists one stable non-run equilibrium. However, as described in section 2.4, any informational shock which causes the expectation of a bank run can push \(c_1^2\) above the threshold and into the bank run equilibrium.

### 2.6 Do Banks Exist?

Given that runs can occur at any time for any reason, one might wonder whether a bank will ever be formed at all. As described in section 2.4, a run leaves a portion \(1/r_1\) agents with \(r_1\) units each and the remaining portion of \(1 - 1/r_1\) agents with 0 units each. This point is especially pertinent since the DD model assumes relative risk aversion. Zero consumption can lead to infinitely negative utility as discussed by Huo and Yu (1994) which shows that for some combinations of \(\gamma, t,\) and \(R\) it may be impossible to establish a bank. The converse is that in some cases banks are optimal despite positive probability of a run.

Let us compare social welfare of the all-production economy \(W_{Prod}\) given in equation 2.3 to that of the all-bank economy \(W_{Bank}\) in equation 2.11. As before, \(W_{Bank}\) is given in terms of weighted expected value given some expected probability \(\delta\) that a run will occur.

\(^5\)These calculations are shown in Appendix B.
as shown in 2.18.

\[
W_{\text{Bank}} = (1 - \delta)W_{\text{NoRun}} + \delta W_{\text{Run}}
= (1 - \delta)[tu(c_1^1) + (1 - t)\rho u(c_1^2 + c_2^2)] + \delta[(1/r_1)u(r_1) + (1 - 1/r_1)u(0)] \\
\geq tu(c_1^1) + (1 - t)\rho u(c_1^2 + c_2^2) = W_{\text{Prod}}
\tag{2.18}
\]

Since a bank will only be formed if welfare for an all-bank economy is expected to be greater than or equal to that of the all-production economy, there must be some value \(\delta\) low enough such that \(W_{\text{Bank}} > W_{\text{Prod}}\) in order for agents to invest in the bank. We can use this inequality in equation 2.11 to solve for the threshold probability \(\hat{\delta}\) for which a bank will be formed.

\[
\hat{\delta} = \frac{W_{\text{NoRun}} - W_{\text{Prod}}}{W_{\text{NoRun}} - W_{\text{Run}}}
\tag{2.19}
\]

For all \(\delta \leq \hat{\delta}\), forming a bank creates an ex ante improvement in social welfare. However, this all-bank economy is only a special case of a more general model.

\section{The Multi-Sector Model}

DD (1983, p.405-410) describes a model which includes both banking and production sectors. However, DD (1983) and all subsequent models have focused on the banking sector alone. This section discusses a DD model with both banking and production. Allowing multiple sectors creates a welfare improvement since agents can hedge their risk by investing in multiple sectors. Exchange between producers and bank depositors further increases welfare and helps prevent bank runs since risky bank notes can be traded at a discount for real goods.

The next section describes how allowing investment in multiple sectors creates a Pareto
improvement in social welfare. Section 3.2 discusses exchange in the DD model. Consumption decisions are discussed in section 3.4. Section 3.3 explains the lack of bank runs in equilibrium.

3.1 Welfare in the Multi-Sector Model

Optimal investment in the DD model occurs when capital is divided between production and deposits. DD (1983) note that under an optimal investment strategy, “agents will invest at least some of their wealth in banks even if they anticipate a positive probability of a run” (p.409, emphasis added). This clearly implies that some capital will remain outside of banks as well. However, DD (1983) does not discuss this scenario in detail.

The article begins with the all-production economy to demonstrate the risk diversification benefits provided by banks. It then describes an all-banking economy stating that “for now we will assume all agents are required to deposit initially” (p.409, emphasis added). The language implies that the authors will also discuss the case in which not all agents are depositors, however they never return to this scenario.

In the multi-sector model, capital is divided between banking and production. Each agent invests a portion $\phi$ of his goods in the bank and $(1 - \phi)$ in production. Payoffs for each sector are the same as previously described, $c_1 = 1$ and $c_2 = R$ for production and $c_1 = r_1$ and $c_2 = r_2(c_1)$ for banking. In the case of a bank run, payoffs to the banking sector are $c_1 = r_1$ with probability $1/r_1$ and $c_1 = 0$ with probability $1 - 1/r_1$.

We calculate ex ante social welfare as the weighted sum of expected utilities for each agent type in three potential states. These are shown in figure 4 along with the probability for each state. When there is no bank run, type 1 agents receive the expected value of $c_1^1 = \phi r_1 + (1 - \phi)(1)$, and type 2 agents receives $c_2^2 = \phi r_2 + (1 - \phi)R$. In the case of a bank run, type 1 agents have probability $1/r_1$ chance of receiving $c_1^1 = \phi r_1 + (1 - \phi)(1)$ and
Agent utility (W\textsubscript{MS})

No run
\((1 - \delta)\)

\begin{align*}
\text{Type 1} & : \phi r_1 + (1 - \phi)(1) \\
\text{Type 2} & : \phi r_2 + (1 - \phi)R
\end{align*}

Run
\((\delta)\)

\begin{align*}
\text{Deposit redeemed} & : (1/r_1) \\
\text{Deposit not redeemed} & : (1 - 1/r_1)
\end{align*}

\begin{align*}
\text{Type 1} & : \phi(0) + (1 - \phi)(1) \\
\text{Type 2} & : \phi(0) + (1 - \phi)R
\end{align*}

\begin{align*}
\text{Type 1} & : \phi r_1 + (1 - \phi)(1) \\
\text{Type 2} & : \phi r_2 + (1 - \phi)R
\end{align*}

Figure 4: Probability-weighted expected value of social welfare.

Let us compare this social welfare function to the other previously discussed welfare functions using three values of \(\delta\). First, notice that if agents know there will be a bank run

\[
W_{MS} = (1 - \delta)\left[ tu(\phi r_1 + (1 - \phi)(1)) + (1 - t)\rho u(\phi r_2 + (1 - \phi)R) \right]
+ \delta t\left[ (1/r_1)u(\phi r_1 + (1 - \phi)(1)) + (1 - 1/r_1)u(\phi(0) + (1 - \phi)(1)) \right]
+ \delta(1 - t)\rho\left[ (1/r_1)u(\phi r_1 + (1 - \phi)R) + (1 - 1/r_1)u(\phi(0) + (1 - \phi)R) \right]
\] (3.1)

probability \((1 - 1/r_1)\) of receiving \(c_1^1 = \phi(0) + (1 - \phi)(1)\). Type 2 agents have probability \(1/r_1\) chance of receiving \(c_1^2 = \phi r_1 + (1 - \phi)R\) and probability \((1 - 1/r_1)\) of receiving \(c_2^1 = \phi(0) + (1 - \phi)R\). Social welfare is the weighted sum of the potential states as calculated in equation 3.1.
(\delta = 1), then no capital will be invested in the bank (\phi = 0), so

\[ W_{MS} = tu(c_1^1) + (1 - t)\rho u(c_2^2) = W_{Prod} \quad (3.2) \]

Similarly, if there are no bank runs (\delta = 0), then all capital will be invested in the bank (\phi = 1), so

\[ W_{MS} = (1 - \delta)[tu(c_1^1) + (1 - t)\rho u(c_2^2)] + \delta[(1/r_1)u(r_1) + (1 - 1/r_1)u(0)] = W_{Bank} \quad (3.3) \]

However, for all values 0 < \phi < 1 we can see that the multi-sector utility is not simply the sum of utility in each sector. For example, the no-run type 1 utility is \phi u(r_1) for banking and (1 - \phi)u(1) for production. However, the multi-sector value of \( u(\phi(r_1) + (1 - \phi)(1)) \) is greater due to relative risk aversion in the utility function. This is true of all states and agent types. Therefore, we know that for all 0 < \phi < 1, \( W_{MS} \) is greater than the linear combination of \( W_{Bank} \) and \( W_{Prod} \).

\[ W_{MS} > (1 - \phi)W_{Prod} + \phi W_{Bank} \quad (3.4) \]

Considering these three cases, only the last is consistent with our prior assumptions. The probability of a bank run was assumed to be positive (\delta > 0), so equation 3.3 can be ignored. Conversely, if the probability of a run is high, then no bank will ever be established, so 3.2 is not relevant. Thus, we can see that social welfare is improved for any value of 0 < \phi < 1 as shown in equation 3.4. The ability to invest in both banking and production sectors creates an ex ante welfare improvement for all agents.\(^6\)

Note that this ex ante welfare improvement is the result of diversification alone. Since utility exhibits relative risk aversion, all agents prefer to invest in goods and production

\(^6\)A complete proof is given in Appendix C.
in order to hedge the risk of becoming impatient. Further gains may be created through trade in $T = 1$ once agents have discovered their utility preferences.

### 3.2 Is Trade Allowed?

Exchange between producers and depositors is clearly described in DD (1983). The authors compare deposit contracts to market exchange and point out cases where trade will not be advantageous. Thus, analysis of a multi-sector economy with exchange is not an extension of the DD model but simply a case within the original model that has not been previously examined.

DD assume tradability of goods and deposit contracts. Trade in consumption goods is clearly allowed as DD predict that “There will be trade in claims on goods for consumption at $T = 1$ and at $T = 2$” (p.406). Regarding deposit contracts, they write that “illiquidity is a property of the financial assets in the economy in our model, even though they are traded in a competitive market” (p.403, emphasis added). The original specification of the model assumes “a competitive market in claims on future goods” (p.406), so even if bank notes themselves were not tradable, agents could transact in forward contracts with the same payoffs.

Some works such as Selgin (1993, p.348) assume that deposit contracts are “nontransactable” in the standard DD model. However, it appears that the reason exchange is not discussed by DD is not that such trade is prohibited but rather that it is non-optimal in the scenarios analyzed. DD (1983, p.406) notes that no exchange will occur in an all-production economy since each agent already holds his optimal consumption bundle. The same conditions apply to the all-bank economy. Only in an economy of both producers and depositors does an agent have the incentive to trade.
3.3 Bank Runs

In contrast to the all-banking model, bank runs do not form an equilibrium in the multi-sector model. The lack of runs can be shown by describing the value $V(c_1)$ of redemption at $T = 1$ to each agent type as illustrated in figure 5. The fundamental values are shown in figure 5a as functions of total redemptions $c_1 = c^1_1 + c^2_1$. The deposit value is constant at $r_1$ for type 1 agents but is falling for type 2 agents since $r_2$ is a decreasing function of $c_1$.

These are the same lines depicted in figure 2a except for the addition of the new horizontal line at $V = r_2/R$. Since deposits represent $r_2$ worth of goods to type 2 agents, these agents will be willing to buy deposits at any price below $r_2/R$. This new line represents the exchange value of deposits.

Let us explore further the incentives for trade. A type 2 producer holds real goods which are worth $c^2_2 = R$ to himself but only $c^1_1 = 1$ to a type 1 agent. Type 1 depositors hold deposits worth $c^1_1 = r_1$ to themselves but worth $c^2_2 = r_2$ to type 2 agents. At these prices, there will be no exchange. However, if type 2 depositors suspect that $c_1 > 1/r_1 - t$, then they will expect a bank run, and the expected value of deposits will fall to $c_1 = r_0 < r_2/R$. At this price, it become profitable for type 2 producers to trade some portion of their
goods for deposits. For each deposit purchased by a type 2 producer, another deposit can be redeemed by a type 1 agent without fear of a bank run. Therefore, the average expected value of type 2 deposits will lie somewhere between in \( r_1 \) and \( r_2/R \). This value is represented by the dashed line in figure 5b, although the actual expected value will depend on the portions of agents \( t \) and deposits \( \phi \) as well as the values used in the utility function.

Just as the existence of deposit insurance prevents bank runs (DD 1983, p.413-416), so does the potential for exchange. If type 2 depositors know that type 2 producers stand ready to purchase deposits, there is no incentive for an expectations-led run. Type 2 depositors are not afraid of other type 2 depositors running on the bank because they have a more profitable option available through exchange. There is still be a potential for a fundamental bank run if \( c_1 > 1 - t \), but the threat of sunspots becomes irrelevant.

Should we worry about fundamental bank runs? A bank becomes insolvent when its commitments exceed its capital. This occurs in the model when the portion \( t \) of type 1 agents exceeds its expected value \( \tilde{t} \) by some amount. When agents realize that the bank is insolvent, all type 2 depositors run on the bank since they know there will be no capital left for redemption at \( T = 2 \). Yet the misjudgment of \( r_1 \) based on \( \tilde{t} \) is no different than any other business error that causes firm failure. A bank that over-promises should be held responsible for its errors and its capital be reallocated to production or other banks. In this sense, the fundamental run is an efficient mechanism for closing insolvent banks. As White (1999, p.121) describes, “A run on an insolvent bank serves the same function as a bankruptcy proceeding.” Therefore, the result of the multi-sector model, that fundamental runs are allowed but sunspot runs are not, is the optimal set of responses to potential runs.

Exchange also provides information conveyed through the price system. The price of goods in terms of deposits provides an indication to all agents of the portion of impatient type 1 agents in the population. This public information may enable other efficient mech-
anisms for preventing runs. DD (1983, p.411-412) demonstrates that suspension of deposit redemption can induce optimal levels of consumption only if the portion \( t \) of type 1 agents is publicly available, which it is not. However, in the multi-sector model banks may be able to implement optimal suspension contracts based on market prices since the portion of type 1 agents may be derived from the price.

### 3.4 Multi-Sector Consumption

Actual consumption in the multi-sector model can be optimized as of time \( T = 1 \). As in section 2.5, \( c_2 \) is a non-linear function of \( c_1 \). Social welfare is the same as in the production economy. Unlike the all-bank optimization, we include some type 1 consumers who may wish to participate in trade. Specifically, we include the portion of \( t(1 - \phi) \) of type 1 agents who invested in production. These agents seek to maximize period 1 consumption and may be able to improve upon their endowment of 1 unit of goods from production. Conversely, type 1 depositors are included as a constant term since these agents already hold their optimum consumption bundle of \( r_1 \) and are unlikely to improve upon that outcome.

\[
W_{\text{Bank}} = t\phi u(r_1) + t(1 - \phi)u(c_1^1) + (1 - t)\rho u(c_1^2 + c_2^2) \tag{3.5}
\]

The resource constraint 3.6 is a combination of the production constraint from equation 2.2 and the banking constraint from equation 2.14 allocated according to \( \phi \).

\[
c_2 = \begin{cases} 
(1 - \phi)R[1 - (1 - \phi)c_1^2] + \phi \frac{R(1 - r_1(t + c_1^2))}{1 - \rho c_1^2} & \text{for } c_1^2 \geq 1 - t \\
0 & \text{for } c_1^2 > 1 - t
\end{cases} \tag{3.6}
\]

This equation describes consumption in period 2 from banking and production. The portion \( \phi \) of capital allocated to banking is depleted by \( 1 - r_1(t + c_1^2) \) redemptions in time \( T = 1 \),
then grows by \( R \), and will be apportioned to \( 1 - t - \phi c_1^2 \) agents at \( T = 2 \). The portion \( 1 - \phi \) of goods allocated to production are depleted only by the type 1 agents \( 1 - (1 - \phi)c_1^2 \).

Bank capital is fully depleted when \( c_1^2 \) reaches \( 1 - t \). This limit is the full amount of capital in the economy excluding the portion \( t\phi \) of type 1 depositors who did not engage in trade.

Optimizing social welfare in equation 3.5 according to the resource constraint 3.6, we find the optimal consumption set in equations 3.7 and 3.8:

\[
c_1^2 = 1 - t \pm \sqrt{\frac{\phi R(r_1 - 1)(1 - t)}{1 - (1 - \phi)R}} \tag{3.7}
\]

\[
c_2^2 = \frac{1 - r}{r - 1} \left(1 + \sqrt{\frac{\phi R(r_1 - 1)(1 - t)}{1 - (1 - \phi)R}}\right) \tag{3.8}
\]

These consumption levels can be seen graphically in figure 6. Like figure 3, the tradeoff between consumption in times \( T = 1 \) and \( T = 2 \) is non-linear. In this case, the y-intercept of \( c_2^2 \) lies between \( r_2 \) and \( R \) since the portion \( \phi \) of capital deposited in the bank is positive but less than 1. However, unlike figure 3, the constraint does not cross the x-axis at \( c_1^2 = 1/r - t \) but instead crosses at \( c_1^2 = 1 - t \) just like the all-production constraint in 1. This is because deposits do not default prematurely but always retain some positive value through trade until the point of consumption \( c_1^2 = 1 - t \) where the bank faces default. Utility for type 1 depositors is maximized by consuming all of their goods in \( T = 1 \) as seen in 6a, while type 2 depositors choose some optimal combination of \( c_1^2 + c_2^2 < 1/r_1 - t \), we know that there is one stable equilibrium with no bank runs and no bank run equilibrium before \( c_1^2 = 1 - t \) where the bank becomes insolvent.

\[\text{These calculations are shown in Appendix D.}\]
4 Historical Evidence

The history of banking in the United States before the Federal Reserve indicates that the multi-sector DD model is more analogous to the real world than the single-sector models. Section 4.1 describes how risky bank notes and deposits were often traded at a discount. Section 4.2 shows that bank runs are based on fundamental factors rather than expectations alone. Section 4.3 emphasizes the important role of information both in the model and in the real world.

4.1 Trading at a Discount

There is ample historical evidence that the deposits, particularly bank notes, of risky banks were traded at a discount to their face values. Gorton (1999, p.36) explains that “a bank note is equivalent to risky debt with maturity equal to the time it takes to return from the particular location of the note holder to the site of the issuing bank.” During the US free banking era (1837-67), the notes issued by private banks were regularly traded at a discount depending upon the expected probability that they would be successfully redeemed. Rolnick and Weber (1988, p.33) provides evidence from this period, stating that “Free bank notes were demanded because they were priced to reflect the expected value
of their backing.” Mullineaux (1987) and Calomiris and Kahn (1996) provide additional evidence of note discounting in the Suffolk banking system.

Some bank notes carried an “option clause” which allowed the bank temporary suspension of convertibility. During periods of suspension, bank notes continued to circulate as currency but were often traded at a discount. The possibility of optimal suspension is discussed by DD (1983, p.410-411), Dowd (1992), Diamond and Rajan (2001), and Peck and Shell (2003). Commenting on the suspended convertability of the greenback from 1862 to 1878, Calomiris (1988, p.190) notes that expectations of future redemptions “play a vital role in determining the exchange rate” of greenbacks to gold. Calomiris and Schweikart (1991) analyze the Panic of 1857 and find that note discounting varied widely across states depending on regulation and clearing house activities. Gorton (1985, p.280-282) and Calomiris and Gorton (1991, p.117-119) describe how multiple banks sometimes coordinated suspensions until the weaker banks could rebuild their capital bases. White (1984, p.26-30) and Dowd (1988, p.325) provide further evidence from Scottish banking of discounted note trading in times of suspension.

During the pre-Fed period there was some a danger of “wildcat” banks opening long enough to issue notes but closing again before their notes could be redeemed. Rockoff (1985, p.887) shows that a discount was often applied to the notes of new banks entering the market which reflected their level of risk. This system was generally effective at limiting the danger of wildcat banking since “market participants could discipline banks by pricing factors that affected risk and via the contractual redemption option” (Gorton 1999, p.61). As a bank’s reputation was successfully established, the price of its notes increased to approach is face value. Gorton (1996) confirms that reputation mechanism caused both the bonds and notes of risky banks to be traded at a discount, noting that “Redemption and reputation, combined with public and private restrictions on risk taking that limited
the degree of adverse selection, explain the success of the Free Banking Era” (Gorton 1996, p.386).

4.2 Fundamentals or Sunspots?

In the DD model, economy-wide bank runs can occur for non-economic reasons which have nothing to do with the bank’s fundamental solvency. However, in reality, sunspots do not cause runs. Even during a financial panic, insolvent banks are more likely to be run upon than solvent banks.

Articles such as Ennis and Keister (2010) cite the bank runs during the Great Depression and panics in the national banking era as evidence of economy-wide runs. However, Gorton (1988) examines this period and finds that bank runs are not caused by sun spots but by bank fundamentals. The study disputes the view that panics “are random events, perhaps self-confirming equilibria in settings with multiple equilibria, caused by shifts in the beliefs of agents which are unrelated to the real economy” and maintains instead that “variables predicting deposit riskiness cause panics just as such movements would be used to price such risk at all other times” (Gorton 1988, p.751). Champ, Smith, and Williamson (1989) and Wicker (2000) show that pre-Fed bank runs tended to occur in heavily regulated regions, indicating that runs were related to fundamental issues rather than wide spread panics. Champ, Freeman, and Weber (1999) suggests that bank runs were often caused by variation in redemption costs, again indicating a fundamental issue rather than a self-fulfilling prophecy.

Other works show that the danger of economy-wide bank runs is often attributable to the central bank rather than being inherent to deposit banking. Gorton (1985, p.222) states that “The number of banks that failed during the thirties was roughly twenty-five times what it would have been had the pre-Federal Reserve System institutions been in
place.” Rolnick and Weber (1986, p.4) notes that pre-Fed bank runs tended not to cause contagion. Selgin, Lastrapes, and White (forthcoming, p.22-25) describes how bank runs actually increased in the early years of the Federal Reserve. It was FDIC deposit insurance rather than the Fed that ended the incidence of banking panics and runs.

4.3 Information and Exchange

The primary source of instability in the DD model is the lack of information. Agents run on the bank when they have insufficient information about the safety of their deposits, and banks cannot prevent runs because they lack information regarding the preferences of their depositors. However, in the real world there are practical solutions to these information asymmetries. Banks provide information about their financial condition, customers can receive verification for their claims, and the price system provides information on both bank condition and consumer preference.

Sunspot bank runs occur in the DD model because agents lack information about the safety of their deposits. However, banks developed mechanisms to share information so that this issue was not a problem. Some banks published their capital levels while others announced their relationships with clearing houses and other lenders. Similarly, customers can have their personal and financial information verified through third parties.

Markets reveal information about the bank through the price system. As described in section 4.1, there are many historical example of notes from risky banks trading at a discount. Discounted note trading was especially important during times of suspension. DD (1983, p.410-411) examines suspension of convertibility as one method for preventing runs. In the all-bank version of the model there is insufficient information upon which banks can base their decision to suspend. However, in the multi-sector version, the price can fulfill this function.
It is often claimed that the DD model illustrates the inherent instability of deposit banking. However, instability in the model originates in restrictions on information and exchange that occur only in certain cases of the model. When studied as part of a larger economy, banking provides a welfare improvement without creating instability. This conclusion is consistent with historical evidence. The multi-sector version of the DD model can be used to study the effectiveness of information and exchange in preventing runs.

5 Conclusions

The generalized Diamond-Dybvig model shows that banking is not inherently unstable when considered as part of a larger economy. The exchange of goods and bank deposit contracts can prevent bank runs on solvent banks. It is curious that the dozens of articles extending the DD model have failed to discuss this scenario. If exchange alone is sufficient to prevent bank runs, the conclusions of some of these works may need to be reconsidered.

Exchange in the DD model opens the door to other preventative solutions to bank runs. First, exchange creates prices which may reveal information about a bank’s financial position. DD (1983, pp.403, 406) states that the concealment of private information prevents competitive markets from providing optimal liquidity. However, information conveyed through the price system might be used as a basis for contingent contracts that allow competitive markets to achieve this optimum. Second, the simultaneous existence of goods and deposit contracts enables trade of deposit contracts in times of suspension. This is consistent with historical evidence and can help prevent runs as described in Selgin (1993). Third, banks may be able to provide the same payoffs as government deposit insurance since risky deposit contracts will trade at a discount in the market. These are but a few examples of ways in which exchange between producers and depositors enables more realistic solutions to be incorporated into the DD model.
This version of the DD model can also be used to study the importance of exchange in inhibiting bank runs. One implication of the model is that restricting the exchange of deposit contracts may actually cause bank runs (or at least handicap the market’s ability to deter them). Historical evidence from the nineteenth and early twentieth centuries supports this notion. The DD model can be used to analyze the degree to which government intervention may have been the cause rather than deterrent of bank runs.
A Optimal Consumption

Let us use the same model set forth in section 2. The production technology has payoffs:

\[
\begin{array}{ccc}
T=0 & T=1 & T=2 \\
-1 & 0 & R \\
1 & 0 & 0
\end{array}
\]

Each agent is endowed with 1 unit of capital which he is required to invest at \( T = 0 \). Payoffs are a nonnegative linear combination of 1 at \( T = 1 \) and \( R \) at \( T = 2 \). The tradeoff between consumption in periods \( T = 1 \) and \( T = 2 \) forms the resource constraint.

\[ c_2 = R(1 - c_1) \]  
(A.1)

Agents are of two types, either impatient \( \Theta = 1 \) or patient \( \Theta = 2 \). The notation \( c_\Theta^t \) represents consumption by agent type \( \Theta \) in time \( t \). The utility function given in equation A.2 is dependent on agent type \( \Theta \) with \( 0 < \rho < 1 \), \( 0 < \gamma < 1 \), and \( c_\Theta^T \geq 0 \).

\[
u(c_1, c_2, \Theta) = \begin{cases} 
\frac{(c_1)^{1-\gamma}}{1-\gamma} & \text{if } \Theta = 1 \\
\frac{\rho(c_1^2 + c_2^{1-\gamma})}{1-\gamma} & \text{if } \Theta = 2 
\end{cases}
\]  
(A.2)

Additionally, we assume that agents would always prefer to be patient and consume at \( T = 2 \). Therefore, we know that \( u(c_2) > u(c_1) \).

The proportion \( t \in (0, 1) \) of type 1 agents in the population of \( N \) agents is calculated in equation A.3 and is assumed (for now) to be known to all agents.

\[ t = \frac{\Sigma_1^N (2 - \Theta)}{N} \]  
(A.3)

30
Total social utility is the weighted sum of expected utilities for all agents.

\[ V = tu(c^1_1) + (1 - t)\rho u(c^2_1 + c^2_2) \]  

(A.4)

Social utility is subject to the budget constraint from A.1 which can be re-written to solve for \( c^2_2 \) in terms of the variables \( c^1_1 \) and \( c^2_1 \) and constants \( t \) and \( R \).

\[ c^2_2 = \frac{R(1 - tc^1_1 + (1 - t)c^2_2)}{(1 - t)} \]  

(A.5)

**Proposition:** The socially optimum levels of consumption \( \hat{c}^1_1, \hat{c}^1_2, \hat{c}^2_1, \) and \( \hat{c}^2_2 \) for agent types 1 and 2 in periods \( T = 1 \) and \( T = 2 \) (calculated ex ante as of time \( T = 0 \)) are not equal to the consumption levels of independent agents \( c^1_1 = 1, c^2_2 = R, c^1_2 = c^2_1 = 0 \).

First, we can see from the utility functions in equation A.2 that \( \hat{c}^2_2 = 0 \) since type 1 agents get no utility from time 2 consumption. The other optimum consumption values can be found according to the Lagrangian function in A.6.

\[ L = tu(c^1_1) + (1 - t)\rho u(c^2_1 + c^2_2) + \lambda(R[1 - tc^1_1 - (1 - t)c^2_1] - (1 - t)c^2_2) \]  

(A.6)

First order conditions are:

\[ \frac{\partial L}{\partial c^1_1} = tu'(c^1_1) + \lambda R = 0 \]  

(A.7)

\[ \frac{\partial L}{\partial c^2_1} = (1 - t)\rho u'(c^2_1 + c^2_2) + \lambda R(1 - t) = 0 \]  

(A.8)

\[ \frac{\partial L}{\partial c^2_2} = (1 - t)\rho u'(c^2_1 + c^2_2) + \lambda(1 - t) = 0 \]  

(A.9)

\[ \frac{\partial L}{\partial \lambda} = R[1 - tc^1_1 - (1 - t)c^2_1] - (1 - t)c^2_2 = 0 \]  

(A.10)
Comparing $\frac{\partial L}{\partial c_1}$ and $\frac{\partial L}{\partial c_2}$ we find that they have equal marginal benefit $\rho u'(c_1^2 + c_2^2)$, however $\frac{\partial L}{\partial c_1}$ has a higher marginal cost $R(1 - t) > (1 - t)$. Agents will, therefore, choose the minimum amount of $c_1^2 = 0$. Equations A.4 and A.6 can then be rearranged into A.11 and A.12 respectively.

\[-\lambda = \frac{u'(c_1^1)}{R}\]  
\[-\lambda = \rho u'(0 + c_2^2)\]

These equations can be combined into equation A.13 and rearranged to form A.14.

\[u'(c_2^2) = \frac{u'(c_1^1)}{\rho R}\]  
\[c_2^2 = (\rho R)^{1/\gamma}c_1^1\]

This version of $c_2^2$ given in A.14 can now be inserted back into $\frac{\partial L}{\partial \lambda}$ into A.10.

\[0 = R[1 - tc_1^1 - (1 - t)(0)] - (1 - t)(\rho R)^{1/\gamma}c_1^1\]

By rearranging A.15, we solve for optimal consumption of type 1 agents in $T = 1$.

\[\hat{c}_1^1 = \frac{R}{tR + (1 - t)(\rho R)^{1/\gamma}}\]

Now inserting equation A.16 back into equation A.14, we can solve for optimal consumption at $T = 2$.

\[\hat{c}_2^2 = \frac{\rho R^{1/\gamma}}{tR + (1 - t)(\rho R)^{1/\gamma}}\]

The final step of this proof is to show that the socially optimal levels of consumption are not equal to the private consumption levels $\hat{c}_1^1 \neq 1$ and $\hat{c}_2^2 \neq R$. To prove this for $\hat{c}_1^1$, we
recall the assumptions that \( 0 < \rho < 1 \) and \( 0 < \gamma < 1 \) imply the following inequalities.

\[
R > \rho R > (\rho R)^{1/\gamma}
\]

Therefore the numerator of A.16 must be larger than the denominator.

\[
R = tR + (1-t)R > tR + (1-t)(\rho R)^{1/\gamma}
\]

Comparing this to equation A.16, we see that \( \hat{c}_1 > 1 \).

\[
\hat{c}_1 = \frac{R}{tR + (1-t)(\rho R)^{1/\gamma}} > \frac{R}{tR + (1-t)R} = 1
\]

To get \( c_2^2 \) from \( c_1^1 \), we multiply by \( (\rho R)^{1/\gamma} \) as in equation A.14. Performing this on equation A.20 and again using the fact that \( (\rho R)^{1/\gamma} < R \), we see that \( \hat{c}_2^2 < R \).

\[
\hat{c}_2^2 = \frac{R(\rho R)^{1/\gamma}}{tR + (1-t)(\rho R)^{1/\gamma}} < R \left( \frac{(\rho R)^{1/\gamma}}{t(\rho R)^{1/\gamma} + (1-t)(\rho R)^{1/\gamma}} \right) = R(1) = R
\]

### B Actual Consumption in the All-Bank Economy

This section will calculate the actual ex post incentives and consumption as of period \( T = 1 \) as described in section 2.3. We show that optimal ex post consumption values do not match the ex ante expected optima.

We assume the same technology and preferences described in appendix A but with a few changes. First, the portion \( t \) of type 1 agents was known to all agents. It is now assumed to be random and stochastic. Instead, there is some expectation \( \tilde{t} \) that is shared by all
agents at $T = 0$. Agents discover their own type at $T = 1$, but this information is not publicly observable. Therefore, agents cannot tell whether the actual value of $t$ is higher, lower, or equal to the expected value $\tilde{t}$.

As demonstrated in appendix A, agents cannot achieve the socially optimal levels of consumption through private action alone. Let us now assume that agents engage in the group action of creating a bank at time $T = 0$. All agents deposit their 1 unit of capital into the bank. They can withdraw some amount $r_1$ at time $T = 1$. The remaining capital will be divided among the patient agents who receive $r_2$ at $T = 2$ with $1 < r_1 < r_2 < R$. The values of $r_1$ and $r_2$ are set at the expected optima of $\hat{c}_1$ and $\hat{c}_2$ calculated from $\tilde{t}$.

$$r_1 = \hat{c}_1^1 = \frac{R}{tR + (1 - \tilde{t}) (\rho R)^{1/\gamma}}$$  \hspace{1cm} (B.1)$$

$$r_2 = \hat{c}_2^2 = \frac{R (\rho R)^{1/\gamma}}{tR + (1 - \tilde{t}) (\rho R)^{1/\gamma}}$$ \hspace{1cm} (B.2)

These new potential payoffs change the agents’ resource constraint. They now have the option of receiving $r_1$ at $T = 1$ or some portion of the remaining capital $r_2$ at $T = 2$. Although the expected value of $r_2$ is the optimal consumption level $r_2 = \hat{c}_2^2$, the actual value of $r_2$ is a function of $c_1^1 + c_2^1$ as shown in equation B.3.

$$r_2 = \frac{R (1 - r_1 (c_1^1 + c_2^1))}{1 - c_1^1 - c_2^1}$$ \hspace{1cm} (B.3)

The important difference between this constraint and equation A.1 used in the previous section is that B.3 is nonlinear. We can see that not all values of $c_2^1$ will fit the constraint that $c_2^2 > c_1^1$. In fact, $r_2$ can be negative or undefined for some values of $c_1^1 + c_2^1$, but these values are not feasible since no type 2 agent will choose negative consumption. Since type
2 agents have the option of consuming at time $T = 1$ or $T = 2$, type 2 agents will choose whichever value is higher between $r_1$ and $r_2$. For any values $r_2 < r_1$, $c_2^2$ goes to zero. In order to satisfy $c_2^2 > c_1^1$, we assume that equation B.3 holds only for values of $c_1^1 + c_2^2 < 1/r_1$. Otherwise, $r_2 = 0$.

The social utility function is the same as before.

$$W_{Bank} = tu(c_1^1) + (1-t)\rho u(c_1^2 + c_2^2)$$ (B.4)

**Proposition:** When agents invest all capital in the bank, their actual ex post levels of consumption $\bar{c}_1^1, \bar{c}_1^2, \bar{c}_2^1$, and $\bar{c}_2^2$ (calculated at $T = 1$) are not equal to the ex ante expected optima $\hat{c}_1^1, \hat{c}_1^2, \hat{c}_2^1$, and $\hat{c}_2^2$ (calculated at $T = 0$).

First, we can see that as before that $c_1^2$ is not included in the utility function, so $\bar{c}_1^2 = 0$. We can therefore conclude that each type 1 agent consumes $c_1^1 = r_1$, and the total consumption by all $t$ type 1 agents is $tr_1$. We can, thus, rewrite our utility equation to solve only for $c_1^2$ and $c_2^2$.

$$W_{Bank} = tu(r_1) + (1-t)\rho u(c_1^2 + c_2^2)$$ (B.5)

The resource constraint is given in B.6.

$$c_2^2 = \begin{cases} 
\frac{R(1-r_1(t+c_1^1))}{k-t-c_1^1} & \text{for } c_1^2 < 1/r_1 - t \\
0 & \text{for } c_1^2 \geq 1/r_1 - t 
\end{cases}$$ (B.6)

To optimize $W_{Bank}$, we use the Lagrangian function in B.7.

$$L = tu(r_1) + (1-t)\rho u(c_1^2 + c_2^2) + \lambda \left( \frac{R(1-r_1(t+c_1^1))}{k-t-c_1^1} - c_2^2 \right)$$ (B.7)
First order conditions are:

\[
\frac{\partial L}{\partial c_1^2} = (1 - t)\rho u' (c_1^2 + c_2^2) + \lambda \left( \frac{-Rr_1(1 - t - c_1^2) + R(1 - r_1(t - c_1^2))}{(1 - t - c_1^2)^2} \right) = 0 \quad (B.8)
\]

\[
\frac{\partial L}{\partial c_2^2} = (1 - t)\rho u'(c_1^2 + c_2^2) - \lambda = 0 \quad (B.9)
\]

\[
\frac{\partial L}{\partial \lambda} = \frac{R(1 - r_1(t + c_1^2))}{1 - t - c_1^2} - c_2^2 = 0 \quad (B.10)
\]

Equations B.8 and B.10 can then be combined into B.11.

\[
0 = \lambda + \lambda \left( \frac{-Rr_1(1 - t - c_1^2) + R(1 - r_1(t - c_1^2))}{(1 - t - c_1^2)^2} \right) \quad (B.11)
\]

Dividing both sides by \( \lambda \) and multiplying by \( (1 - t - c_1^2)^2 \), we get B.12.

\[
0 = (1 - t - c_1^2)^2 - Rr_1(1 - t - c_1^2) + R(1 - r_1(t - c_1^2)) \quad (B.12)
\]

This can be expanded to B.13 and simplified to B.14.

\[
0 = (1 - t - c_1^2 - t - t^2 + tc_1^2 - c_1^2 + tc_1^2 + (c_1^2)^2) - (Rr_1 + Rr_1 t + Rr_1 c_1^2) + (R - Rr_1 t - Rr_1 c_1^2) \quad (B.13)
\]

\[
0 = (c_1^2)^2 + (2t - 2)c_1^2 + (1 - 2t + t^2 - Rr_1 + R) \quad (B.14)
\]

Using the quadratic formula to solve for \( c_1^2 \), we find B.15 which can be simplified to B.16.

\[
c_1^2 = \frac{-2t + 2 \pm \sqrt{(2t - 2)^2 - 4(1 - 2t + t^2 - Rr_1 + R)}}{2} \quad (B.15)
\]

\[
c_1^2 = 1 - t \pm \sqrt{R(r_1 - 1)} \quad (B.16)
\]
This gives us two possible cases for $c_1^2$. Inserting these potential values into our resource constraint from equation B.6, we find that the value of $c_1^2 = 1 - t + \sqrt{R(r_1 - 1)}$ violates the assumption of $c_2^2 \leq R$ as shown in equation B.17.

$$
c_2^2 = R \left( \frac{1 - r_1 t - r_1(1 - t + \sqrt{R(r_1 - 1)})}{1 - t - (1 - t + \sqrt{R(r_1 - 1)})} \right) = R \left( \frac{1 - r_1 t - r_1 + r_1 t - r_1 \sqrt{R(r_1 - 1)}}{1 - t - 1 + t - \sqrt{R(r_1 - 1)}} \right)
$$

$$
= R \left( \frac{1 - r_1 - r_1 \sqrt{R(r_1 - 1)}}{-\sqrt{R(r_1 - 1)}} \right) = R \left( \frac{r_1 - 1}{-\sqrt{R(r_1 - 1)}} + \frac{-r_1 \sqrt{R(r_1 - 1)}}{-\sqrt{R(r_1 - 1)}} \right)
$$

$$
= R \left( \frac{-(r_1 - 1)}{-\sqrt{R(r_1 - 1)}} + r_1 \right) = R \left( r_1 + \frac{(r_1 - 1)}{\sqrt{R(r_1 - 1)}} \right)
$$

$$
> R(r_1) > R(1) = R
$$

(B.17)

Alternatively, we see that $c_2^2 = 1 - t - \sqrt{R(r_1 - 1)}$ in equation B.18 fits the assumption that $c_2^2 \leq R$.

$$
c_2^2 = R \left( \frac{1 - r_1 t - r_1(1 - t - \sqrt{R(r_1 - 1)})}{1 - t - (1 - t + \sqrt{R(r_1 - 1)})} \right) = R \left( \frac{1 - r_1 t - r_1 + r_1 t - r_1 \sqrt{R(r_1 - 1)}}{1 - t - 1 + t + \sqrt{R(r_1 - 1)}} \right)
$$

$$
= R \left( \frac{1 - r_1 + r_1 \sqrt{R(r_1 - 1)}}{\sqrt{R(r_1 - 1)}} \right) = R \left( \frac{1 - r_1}{\sqrt{R(r_1 - 1)}} + \frac{r_1 \sqrt{R(r_1 - 1)}}{\sqrt{R(r_1 - 1)}} \right)
$$

$$
= R \left( \frac{-(r_1 - 1)}{\sqrt{R(r_1 - 1)}} + r_1 \right) = R \left( r_1 - \frac{(r_1 - 1)}{\sqrt{R(r_1 - 1)}} \right) = R \left( r_1 - \sqrt{\frac{(r_1 - 1)}{R}} \right)
$$

$$
< R(r_1 - (r_1 - 1)) = R(1) = R
$$

(B.18)

We can therefore conclude that the optimal levels of consumption $\bar{c}_1^2$ and $\bar{c}_2^2$ are given in equations B.19 and B.20.

$$
\bar{c}_1^2 = 1 - t - \sqrt{R(r_1 - 1)}
$$

(B.19)

$$
\bar{c}_2^2 = \frac{R - Rr_1(1 - \sqrt{R(r_1 - 1)})}{\sqrt{R(r_1 - 1)}}
$$

(B.20)
We can see that these ex post optimal consumption levels do not match the ex ante expected optima since $\bar{c}_1^2 > 0$, and $\bar{c}_2^2 < r_2$.

C Welfare in the Multi-Sector Economy

This section will show that investment in both banking and production sectors creates a welfare improvement. We begin by restating the social welfare function of the multi-sector economy as described in figure 4 and equation 3.1.

\[
W_{MS} = (1 - \delta)[tu(\phi r_1 + (1 - \phi)(1)) + (1 - t)\rho u(\phi r_2 + (1 - \phi)R)] \\
+ \delta t[(1/r_1)u(\phi r_1 + (1 - \phi)(1)) + (1 - 1/r_1)u(\phi(0) + (1 - \phi)(1))] \\
+ \delta(1 - t)\rho[(1/r_1)u(\phi r_1 + (1 - \phi)R) + (1 - 1/r_1)u(\phi(0) + (1 - \phi)R)] \quad (C.1)
\]

We can see that when the there is zero probability of a bank run ($\delta = 0$), then all capital will be invested in the bank ($\phi = 1$), so

\[
W_{MS} = (1 - \delta)[tu(c_1^1) + (1 - t)\rho u(c_2^1)] + \delta[(1/r_1)u(r_1) + (1 - 1/r_1)u(0)] = W_{Bank} \quad (C.2)
\]

Alternatively, when a bank run is expected ($\delta = 1$), then no capital will be invested in the bank ($\phi = 0$).

\[
W_{MS} = tu(c_1^1) + (1 - t)\rho u(c_2^1) = W_{Prod} \quad (C.3)
\]

However, these scenarios have been ruled out by prior assumptions. Section 2.1 assumed that the probability of a run is positive. Section 2.4 assumed that the probability of a run is small enough that a bank will be formed. Therefore, we need only consider a the case
$0 < \phi < 1$ where positive amounts of capital are invested in both banking and production.

**Proposition:** For an economy with positive probability of a bank run $\delta > 0$, dividing capital between production and banking sectors such that $0 < \phi < 1$ creates higher welfare than investing in either sector alone.

The social utility function C.1 takes advantage of the agents’ relative risk aversion as shown in equation A.2. Therefore, he prefers $u(\phi x + (1-\phi)y)$ over $\phi u(x) + (1-\phi)u(y)$. As shown in figure 4, there are six potential utility outcomes which depend on the agent’s type, his holdings of goods and deposits, and some random probability that the agent will be able to redeem his deposits in the case of a bank run. In each of these cases, the agent is partly hedged since against the possibility of a bank run as demonstrated by the following inequalities:

\[
\begin{align*}
    u(\phi r_1 + (1 - \phi)(1)) &> \phi u(r_1) + (1 - \phi)u(1) \quad \text{(C.4)} \\
    u(\phi r_2 + (1 - \phi)R) &> \phi u(r_2) + (1 - \phi)u(R) \quad \text{(C.5)} \\
    u(\phi r_1 + (1 - \phi)(1)) &> \phi u(r_1 + (1 - \phi)u(1) \quad \text{(C.6)} \\
    u(\phi(0) + (1 - \phi)(1)) &> \phi u(0) + (1 - \phi)u(1) \quad \text{(C.7)} \\
    u(\phi r_1 + (1 - \phi)R) &> \phi u(r_1 + (1 - \phi)u(R) \quad \text{(C.8)} \\
    u(\phi(0) + (1 - \phi)R) &> \phi u(0) + (1 - \phi)u(R) \quad \text{(C.9)}
\end{align*}
\]

Since each of the hedged values is lower than the unhedged values, we know that substituting these into C.1 will give us something lower than $W_{MS}$. We then rearrange these and
simplify to get equation C.13.

\[ W_{MS} > (1 - \delta)[t(\phi u(r_1) + (1 - \phi)u(1)) + (1 - t)\rho(\phi u(r_2) + (1 - \phi)u(R))] \\
+ \delta t[(1/r_1)(\phi u(r_1) + (1 - \phi)u(1)) + (1 - 1/r_1)(\phi u(0) + (1 - \phi)u(1))] \\
+ \delta(1 - t)\rho[(1/r_1)(\phi u(r_1) + (1 - \phi)u(R)) + (1 - 1/r_1)(\phi u(0) + (1 - \phi)u(R))] \\
+ \delta(1 - \phi)(1 - \phi)[t u(1) + (1 - t)\rho u(0)] \\
+ \delta(1 - \phi)(1 - \phi)[t u(r_1) + (1 - r_1)(1 - t)\rho u(0)] \\
+ \delta(1 - \phi)(1 - \phi)[t u(0) + (1 - l_1)(1 - t)\rho u(0)] \\
= (1 - \phi)[t u(1) + (1 - t)\rho u(0)] \\
+ \delta(1 - \phi)(1 - \phi)[t u(r_1) + (1 - r_1)(1 - t)\rho u(0)] \\
+ \delta(1 - \phi)(1 - \phi)[t u(0) + (1 - l_1)(1 - t)\rho u(0)] \\
= (1 - \phi)W_{prod} + \phi W_{Bank} \\
\]

\[ (C.10) \]

\[ (C.11) \]

\[ (C.12) \]

\[ (C.13) \]

**D  Consumption in the Multi-Sector Economy**

This section will calculate the actual ex post incentives and consumption as of period \( T = 1 \) as described in section 3. We assume that preferences described in appendix A and that resource constraints are a combination of the regular technological constraint.
from equation A.1 in section A and the banking constraint from equation B.3 of section B. These constraints are combined according to the portions of capital invested in the production and banking sectors as shown in equation D.1.

\[ c_2 = (1 - \phi)R[1 - (1 - \phi)c_1^2] + \phi \frac{R(1 - r_1(t + c_1^2))}{1 - t - \phi c_1^2} \]  \hspace{1cm} \text{(D.1)}

We assume that each type 1 depositors consume \( c_1^1 = r_1 \) and type 2 depositors divide consumption between \( c_1^2 \) and \( c_2^2 \). The social utility function is the same as before.

\[ W_{\text{Bank}} = tu(r_1) + (1 - t)\rho u(c_1^2 + c_2^2) \]  \hspace{1cm} \text{(D.2)}

To optimize \( c_1^2 \) and \( c_2^2 \), we use the Lagrangian function in D.3.

\[ L = tu(r_1) + (1 - t)\rho u(c_1^2 + c_2^2) + \lambda \left( (1 - \phi)R[1 - (1 - \phi)c_1^2] + \phi \frac{R(1 - r_1(t + c_1^2))}{1 - t - \phi c_1^2} - c_2^2 \right) \]  \hspace{1cm} \text{(D.3)}

First order conditions are:

\[ \frac{\partial L}{\partial c_1^2} = (1 - t)\rho u'(c_1^2 + c_2^2) + \lambda \phi R \left( \frac{-r_1(1 - t - c_1^2) + (1 - r_1(t - c_1^2))}{(1 - t - c_1^2)^2} \right) + \lambda (1 - \phi)R \frac{-1}{1 - t} = 0 \]  \hspace{1cm} \text{(D.4)}

\[ \frac{\partial L}{\partial c_2^2} = (1 - t)\rho u'(c_1^2 + c_2^2) - \lambda = 0 \]  \hspace{1cm} \text{(D.5)}

\[ \frac{\partial L}{\partial \lambda} = (1 - \phi)R[1 - (1 - \phi)c_1^2] + \phi \frac{R(1 - r_1(t + c_1^2))}{1 - t - \phi c_1^2} - c_2^2 = 0 \]  \hspace{1cm} \text{(D.6)}

Equations D.4 and D.6 can then be combined into D.7.

\[ -\lambda = \lambda \phi R \frac{-(r - 1)}{(1 - t - c_1^2)^2} + \lambda (1 - \phi)R \frac{-1}{1 - t} \]  \hspace{1cm} \text{(D.7)}
Dividing both sides by $-\lambda$, we get D.8.

\[ 1 = \phi R \frac{r - 1}{(1 - t - c_1^2)^2} + R \frac{1 - \phi}{1 - t} \]  

(D.8)

Re-arranging and multiplying by $(1 - t - c_1^2)^2$, we get D.9

\[ (1 - R \frac{1 - \phi}{1 - t})(1 - t - c_1^2)^2 = \phi R(r - 1) \]  

(D.9)

Then dividing both sides by $(1 - R \frac{1 - \phi}{1 - t})$, we get equation D.10.

\[ (1 - t - c_1^2)^2 = \frac{\phi R(r - 1)(1 - t)}{1 - R(1 - \phi)} \]  

(D.10)

We take the square root of both sides to get D.11 and re-arranged into D.12.

\[ 1 - t - c_1^2 = \sqrt{\frac{\phi R(r - 1)(1 - t)}{1 - R(1 - \phi)}} \]  

(D.11)

\[ c_1^2 = 1 - t \pm \sqrt{\frac{\phi R(r - 1)(1 - t)}{1 - R(1 - \phi)}} \]  

(D.12)

We know from equation B.18 that in order to satisfy our prior assumptions the square root in equation D.12 must be subtracted rather than added. We can therefore conclude that the optimal levels of consumption $\bar{c}_1^2$ and $\bar{c}_2^2$ are given in equations D.13 and D.14.

\[ \bar{c}_1^2 = 1 - t - \sqrt{\frac{\phi R(r - 1)(1 - t)}{1 - R(1 - \phi)}} \]  

(D.13)

\[ \bar{c}_2^2 = \frac{R - Rr}{\sqrt{\frac{\phi R(r - 1)(1 - t)}{1 - R(1 - \phi)}}} \]  

(D.14)
References


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